Using ABCD Matrices to find the position of a Gaussian beam waist

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Introduction

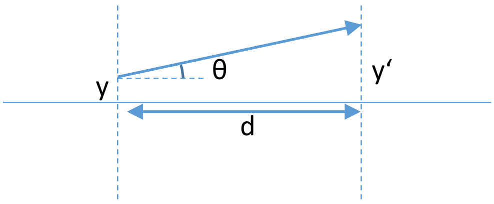
In modern times, the position of beam waists of Gaussian beams are very readily found in optical design software packages but this gives no insight into why or how the beam waist finds itself in that position. The following shows that a pair of equations can determine the position and size of a new beam waist if the position and size of an initial beam waist is known and there is an ABCD matrix representation of the optical components following the initial beam waist. The following sections will introduce the ABCD matrix representation and the two equations. This will be followed by some examples of how the equations can be used.

ABCD Matrices

The ABCD matrix is a simple way of representing paraxial optical components, i.e. where the aberrations introduced by any focusing optical elements is small. This tends to be for beams with small divergence or diameter. These matrices are derived from the representation of a ray of light passing through the optical component. The ray is defined by an initial and final position (y, y’) and an initial and final angle (θ, θ’). A four element matrix can then describe how the new position and angle are derived from the initial values.

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$

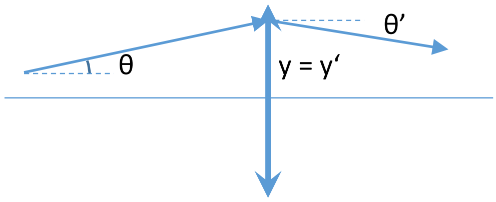
For example, a simple space of length d has a ray diagram like:



In this case, the relation between the input and output positions and angles are described by the following ABCD matrix:

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Similarly, for a perfect thin lens of focal length f, a ray behaves as:



and the ABCD matrix has the following form:

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Beam Waist Equations

If the initial position of the beam waist is known (at z = 0) together with the radius of the Gaussian distribution (ω_0), it is possible to use the ABCD matrices in conjunction with the following two equations to find the position and radius of the new beam waist.

$$BD + ACq^2 = 0 \qquad (1)$$

$$\omega_1^2 = \frac{\omega_0^2}{(AD-BC)}\left[A^2 + \frac{B^2}{q^2}\right] \qquad (2)$$

Equation (1) can be solved to find the new position of the beam waist and equation (2) determines the new beam waist radius. It should be noted that these equation are for the old and new waists both being in air (refractive index = 1). In addition:

$$q = \frac{\pi\omega_0^2}{\lambda}$$

Where λ is the wavelength of the Gaussian beam.

For most systems, the parameter (AD-BC) is equal to 1.

Examples

The following are some examples of how these two equations are used.

Beam Waist Focussed by a Thin Lens

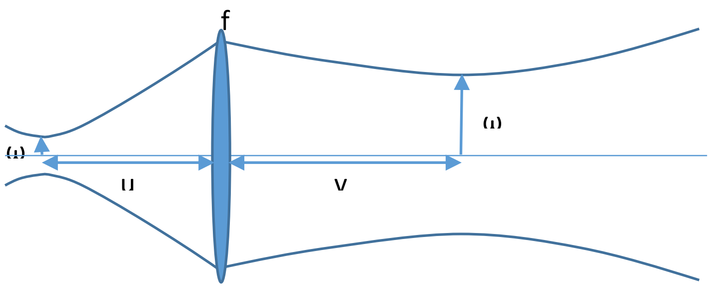


Figure 1 Beam waist formation by a single thin lens.

Figure 1 shows a general imaging system where a beam waist is located a distance u in front of a thin lens of focal length f. The new beam waist is assumed to be a distance v behind the lens. A negative value of v would imply a virtual beam waist on the same side of the lens as the original waist. To generate the final ABCD matrix, the individual matrices representing each section of the system have to be multiplied together. Compared to the order in the figure, the multiplication is performed in reverse order. So, the matrices representing Figure 1 in the required order are:

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$

The ABCD matrix becomes:

$$\begin{pmatrix} 1 - {}^{v}/_{f} & u + v\left(1 - {}^{u}/_{f}\right) \\ -{}^{1}/_{f} & 1 - {}^{u}/_{f} \end{pmatrix}$$

It can be shown that AD-BC = 1.

Substituting this into equation 1, we get:

$$u\left(1 - \frac{u}{f}\right) + v\left(1 - \frac{u}{f}\right)^2 - \frac{1}{f}\left(1 - \frac{v}{f}\right) q^2 = 0$$

And solving for v we get:

$$v = \frac{\frac{q^2}{f} - u\left(1 - \frac{u}{f}\right)}{\frac{q^2}{f^2} + \left(1 - \frac{u}{f}\right)^2} \tag{3}$$

So there is only ever one solution to the position to the beam waist but how big is it?

Substituting the ABCD values into equation (2):

$$\omega_1^2 = \omega_0^2 \left[\left(1 - {}^{v}/_{f}\right)^2 + \frac{1}{q^2}\left(u + v\left(1 - {}^{u}/_{f}\right)\right)^2\right] \tag{4}$$

Using the value of v in this equation, we can generate a value for the new beam waist radius.

<u>Special case 1: u = f</u>

If the first beam waist is positioned on the back focal plane of the lens, then using equation (3), the position of the new beam waist is at v = f. In other words, it is at the front focal plane of the lens.

Substituting these values of u and v into the equation for the beam waist radius, it is found that:

$$\omega_1 = {}^{f}/_{q} = \frac{\lambda f}{\pi \omega_0}$$

This is simply the focal length of the lens multiplied by the divergence of the original Gaussian beam.

Special case 2: u = 0

In [1], an example is given for Gaussian beam propagation where the input beam waist is placed at the same plane as a thin lens of focal length f. Using equation (3), the position of the new beam waist is:

$$v = \frac{f}{1 + \frac{f^2}{q^2}}$$

And similarly from equation (4), the radius of the new beam waist is:

$$\omega_1 = \omega_0 \frac{f/q}{\sqrt{1 + f^2/q^2}}$$

These are identical to the solution given in the reference.

Beam Waist Focussed by a Pair of Thin Lenses

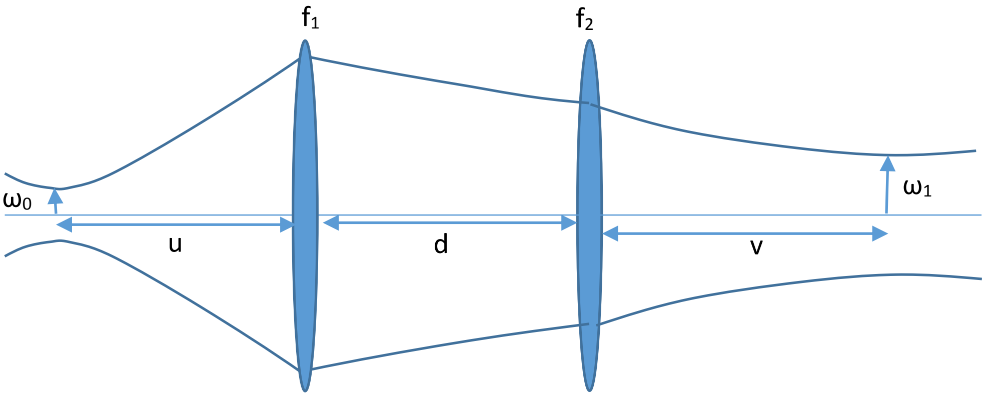


Figure 2 Beam waist formation by a pair of lenses.

Now the matrix equation becomes:

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$

And the ABCD matrix is:

[1] Optical Electronics by A. Yariv, Saunders College Publishing, 4th Edition.

$$\begin{pmatrix} 1-\frac{d}{f_1}-v\left(\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}\right) & u+v-uv\left(\frac{1}{f_1}+\frac{1}{f_2}\right)+d\left(1-\frac{u}{f_1}-\frac{v}{f_2}+\frac{uv}{f_1 f_2}\right) \\ -\frac{1}{f_1}\left(1-\frac{d}{f_2}\right)-\frac{1}{f_2} & 1-\frac{d}{f_2}-u\left(\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}\right) \end{pmatrix}$$

Solving for v and ω_1 using this matrix would generate a general solution for the position and radius of the new Gaussian waist. Rather than doing this, let's consider the special case when u = f_1 and d = f_1 + f_2. This is the commonly used 4f system used to focus and recollimate a laser beam. Substituting these values, the matrix becomes:

$$\begin{pmatrix} -\frac{f_2}{f_1} & f_1 - v\frac{f_1}{f_2} \\ 0 & -\frac{f_1}{f_2} \end{pmatrix}$$

Using equation (1), the position of the new beam waist is v = f_2. In other words, it is at the back focal plane of the second lens. The new waist radius is determined by equation (2) and is:

$$\omega_1 = \omega_0 \frac{f_2}{f_1}$$

The term f_2/f_1 is the linear magnification of the optical system.

In conclusion…

Using two simple equations and a matrix representation of the optical system, it is possible to calculate the new position and radius of a Gaussian beam. Having defined the new beam radius, it is very simple to calculate the beam radius and divergence at other positions in optical system. The specific examples given above are some of the more typically used in practical optical systems. If you found this interesting and would like to know more about what we do, then contact us on info@lumoptica.com.